How Do We Solve Differential Equations?

- If we can spot answer immediately, we can solve by **inspection**
- If the DE doesn't explicitly depend on the dependent variable (and only contains one type of derivative) we can solve by **direct integration**

  e.g $\frac{dy}{dx} = f(x)$ function doesn't depend on y

- For some (but not all) 1st order DEs we can solve by **separation of variables**

  e.g $\frac{dy}{dx} = f(x)\,g(y) \longrightarrow \int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

- For some (but not all) DEs we can use a **substitution**.

Linearity: an operator L is linear if

$$L(ax + by) = aLx + bLy$$

for all scalars a & b and all vectors x & y

- We're interested in operators that act on functions

→ e.g differentiation

Differentiation is a linear operation as

$$\left.\begin{aligned} \frac{d}{dt}\left(af(t)\right) &= a\frac{d}{dt}f(t) \\ \frac{d}{dt}\left(f(t) + g(t)\right) &= \frac{d}{dt}f(t) + \frac{d}{dt}g(t) \end{aligned}\right\} \text{linear properties}$$

similarly, integration : $\int af(t)\,dt = a\int f(t)\,dt$

and expectation : $E(aX) = aE(X)$

we can write $\frac{dx}{dt} = t$ as $\frac{d}{dt}x = t \longrightarrow Lx = t$

linear operator

another example:

$$\frac{dx}{dt} + x = \sin(t) \longrightarrow \boxed{\left(\frac{d}{dt} + 1\right)} x = \sin(t) \longrightarrow Lx = \sin(t)$$

for $\frac{dx}{dt} = x^2$ $\quad (ax)^2 \neq ax^2 \quad \therefore$ is non-linear

to summarise, an ODE is linear if it can be written in the form

$$\boxed{Lx = f(t)}$$

where L is a linear operator

General form of Linear ODE:

General: $a_n(t)\frac{d^n y}{dt^n} + a_{(n-1)}(t)\frac{d^{(n-1)}y}{dt^n} + \dots \; a_1(t)\frac{dy}{dt} + a_0(t)y = f(t)$

e.g 1^{st} order: $\quad a(t)\frac{dx}{dt} + b(t)x = f(t)$

this function can just be a number (e.g 3)

A linear ODE is homogenous if rhs is zero

$$Lx = 0$$

and non-homogenous if rhs is not zero

$$Lx = f(t)$$

Superposition Principle:

If y_1 and y_2 are solutions to the linear homogenous equation $Ly = 0$

$y_{new} = ay_1 + by_2$ is also a solution.

for an n^{th} order linear homogenous OE

$$y = c_1 y_1 + c_2 y_2 \dots + c_n y_n$$

where y's are linearly independent solutions and c's are arbitrary constants

linearly independent solutions cannot be multiplied by constant to get the other

e.g $\sin x$ & $\cos x$ independent

but e^x and $2e^x$ dependent

eg. $$\frac{d^2y}{dx^2} + y = 0$$

2nd order so 2 constants expected

$$\frac{d^2y}{dx^2} = -y$$ 'double derivative of y = minus y'

$\rightarrow$ $y = \sin x$

$y = \cos x$

$$\therefore \underbrace{y = A\sin x + B\cos x}_{\text{general solution}}$$

Constant Coefficient Homogenous Case : (Linear ODE)

Which meet this requirement ?

$$\frac{dx}{dt} = x^2$$ not linear

$$\frac{d^2x}{dt} + tx = 0$$ not constant coefficient

$$\frac{dx}{dt} + 4x = \sin t$$ not homogenous

$$\frac{d^2x}{dt^2} + 4x = 0$$ ✓

Procedure :

1. Classify
2. Use exponential ansatz
3. Find characteristic equation (aka auxillary equation)
4. Find elementary solutions
5. Find general solution

example: $\frac{d^2x}{dt^2} - 4x = 0$ 1. Classify : linear constant homogenous ✓

$x(0) = 1$, $\dot{x}(0) = 2$

2. Ansatz $\rightarrow$ sub $x = e^{\lambda t}$

$\frac{dx}{dt} = \lambda e^{\lambda t}$ and $\frac{d^2x}{dt^2} = \lambda^2 e^{\lambda t}$

Substitute

$\lambda^2 e^{\lambda t} - 4e^{\lambda t} = 0$ $\quad (\lambda^2 - 4) e^{\lambda t} = 0$ $\quad e^{\lambda t} \neq 0$

3. $\lambda^2 - 4 = 0$ $\therefore \lambda = \pm 2$

4. Solutions : $x_1 = e^{2t}$ & $x_2 = e^{-2t}$

5. Can form general solution $x = ae^{2t} + be^{-2t}$

Subbing initial conditions for constants :

$x(0) = 1 \longrightarrow 1 = ae^{2(0)} + be^{-2(0)} \longrightarrow 1 = a + b$

$\dot{x}(0) = 2 \longrightarrow 2 = 2ae^{2(0)} - 2be^{-2(0)} \longrightarrow 2 = 2a - 2b$

$1 = a - b$

Solving simultaneously : $a = 1$, $b = 0$

$\therefore$ Solution $x = e^{2t}$

Pure Imaginary Roots :

Take example of $y'' + 4y = 0$

using ansatz , we get characteristic equation

$\lambda^2 + 4 = 0$

$\lambda^2 = -4$ $\therefore \lambda_1 = 2j$, $\lambda_2 = -2j$

$\rightarrow \lambda = Ae^{2jt} + Be^{-2jt}$

↳ this is imaginary for real t values, whereas we expect an answer to differential equation to be real.

- We want to choose coefficients such that y is real output.
  ↳ if we pick A & B to be complex conjugates of eachother , then

Ae^{2jt} & Be^{-2jt} also are complex conjugates of eachother, and two complex conjugates add to make a real number.

- If we set $A = \alpha + \beta j$ & $B = \alpha - \beta j$

Substituting in : $y = (\alpha + \beta j)e^{2jt} + (\alpha - \beta j)e^{-2jt}$

$$y = \alpha(e^{2jt} + e^{-2jt}) + \beta j(e^{2jt} - e^{-2jt})$$

$$\cos 2t = \frac{e^{2jt} + e^{-2jt}}{2} \qquad \sin 2t = \frac{e^{2jt} - e^{-2jt}}{2j}$$

$\therefore$ $y = 2\alpha \cos 2t + 2\beta j^2 \sin 2t$

$\longrightarrow$ $y = 2\alpha \cos 2t - 2\beta \sin 2t$

as α & β are arbitrary constants, 2α & 2β can be written as constants.

$$y = C\cos 2t + D\sin 2t$$

we can also write in terms of amplitude E and phase shift ϕ:

$y = E\cos(2t + \phi)$

as $E\cos(2t+\phi) = \underbrace{E\cos\phi}_{C}\cos 2t - \underbrace{E\sin\phi}_{D}\sin 2t$

So, if the roots of the auxillary / characteristic equation are pure imaginary so that $\lambda = \pm\omega j$, we can write solution in 3 equivalent ways:

$y = Ae^{j\omega t} + Be^{j\omega t}$, $y = C\cos(\omega t) + D\sin(\omega t)$, $y = E\cos(\omega t + \phi)$

exponential form, cos and sin form, amp. and phase shift form

Complex Roots : (real & imaginary components)

e.g $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 0$

$x = e^{\lambda t}$ $\longrightarrow$ $\lambda^2 + 4\lambda + 5 = 0$

using quad formula : $\lambda = -2 \pm j$

$x_1 = e^{(-2+j)t}$, $x_2 = e^{(-2-j)t}$

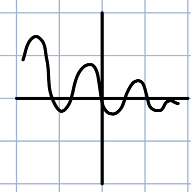
$x = Ae^{(-2+j)t} + Be^{(-2-j)t}$

$x = Ae^{-2t}e^{jt} + Be^{-2t}e^{-jt}$

$x = e^{-2t}(Ae^{jt} + Be^{-jt}) \longrightarrow x = e^{-2t}(C\cos t + D\sin t) \longrightarrow x = Ae^{-2t}\cos(t - \phi)$

decaying exponential

oscillating cos



sketch

General Cases :

Real Roots $x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

Imaginary Roots $x = A\cos\omega t + B\sin\omega t$

Complex Roots $x = e^{-kt}(A\cos\omega t + B\sin\omega t)$

Repeated Roots :

example : $y'' - 6y' + 9y = 0$

characteristic $= \lambda^2 - 6y + 9 = 0$

$(\lambda - 3)^2 = 0$

$\lambda = 3$, repeated root

$y = Ae^{3t} + B$??

Use a substitution of $y = e^{3t}v$

$y' = 3e^{3t}v + e^{3t}v'$

$y'' = 9e^{3t}v + 6e^{3t}v' + e^{3t}v''$

Substituting into DE : $\cancel{9e^{3t}v} + \cancel{6e^{3t}v'} + e^{3t}v'' - \cancel{18e^{3t}v} - \cancel{6e^{3t}v'} + \cancel{9e^{3t}v} = 0$

$e^{3t}v'' = 0$

$e^{3t} \neq 0$

$v'' = 0$

$\therefore v = A + Bt$

$y = e^{3t}v = Ae^{3t} + Bte^{3t}$

one involves regular root

one involves same term but multiplied by t

So, if auxillary equation is a quadratic with λ as a repeated root, the solution is

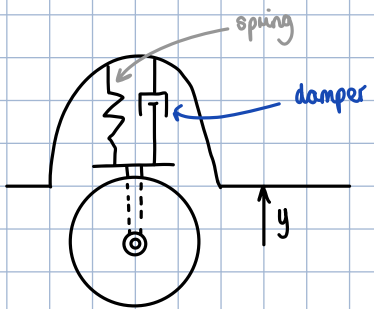
Quadratic : $\boxed{\lambda = Ae^{\lambda t} + Bte^{\lambda t}}$

More generally, a root repeated k times requires all powers of t up to t^{k-1}, therefore if auxillary equation has the root λ repeated k times, the solution is

General : $\boxed{\lambda = C_1 e^{\lambda t} + C_2 t e^{\lambda t} + \ldots + C_k t^{k-1} e^{\lambda t}}$

Damped Simple Harmonic Motion :

example : mass spring damper on car driving over bump



$$m \underbrace{\frac{d^2y}{dt^2}}_{a} = \underbrace{-b\frac{dy}{dt} - ky}$$

act against motion in opposite direction to y
∴ minus

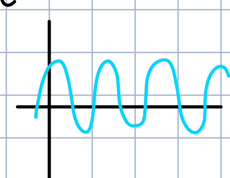
$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \qquad m = 1, \; k = 25$$

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 25y = 0$$

testing different dampening levels : $b = 0$ (no damping)

$$\frac{d^2y}{dt^2} + 25y = 0 \qquad \lambda = \pm 5j \longrightarrow y = A\sin 5t + B\cos 5t$$

$$= C\sin(5t + \phi)$$



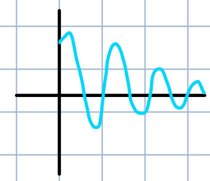
Underdamped

oscillates like sinusoid forever which isn't realistic

$$b = 8 \longrightarrow \frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 25y = 0$$

$$\lambda = -4 \pm 3j \longrightarrow y = Ae^{-4t}\sin(3t + \phi)$$

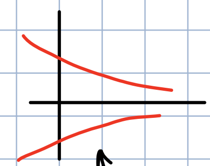
has decay which produces light damping



Underdamped

$b = 26 \longrightarrow \lambda = -25, -1 \longrightarrow y = Ae^{-25t} + Be^{-t}$

both terms are decaying exponentials, no oscillation



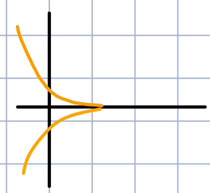
Overdamped

+ve or negative displacements

$b = 10 \longrightarrow \lambda = -5, -5 \longrightarrow y = Ae^{-5t} + Bte^{-5t}$

repeated roots

amplitude reduced at quickest rate



Critical (reaches 0 fastest)